

Optimal detection of sources on a homogeneous & isotropic background

J. L. Sanz¹, D. Herranz^{1,2} and E. Martínez-González¹

(1) Instituto de Física de Cantabria, Fac. de Ciencias, Av. de los Castros s/n,
39005-Santander, Spain

(2) Departamento de Física Moderna, Universidad de Cantabria, 39005-Santander, Spain

Received _____; accepted _____

ABSTRACT

This paper introduces the use of pseudo-filters that optimize the detection/extraction of sources on a background. We assume as a first approach that such sources are described by a spherical (central) profile and that the background is represented by a homogeneous & isotropic random field. We make an n-dimensional treatment making emphasis in astrophysical applications for spectra, images and volumes, for the cases of exponential and Gaussian source profiles and scale-free power spectra to represent the background.

Subject headings: methods: analytical — methods: data analysis

1. Introduction

One of the challenges in the analysis of 1D spectra, 2D images or 3D volumes in Astrophysics and Cosmology is to overcome the problem of separating a localized signal from a background. In particular, we are interested in localized sources with central symmetry and a background that we shall assume with the properties of homogeneity and isotropy and it will be characterized by a power spectrum. Typical cases in the 1D case include: a) the spectra of QSOs, how to detect/extract absorption lines from a noisy spectrum once a profile is assumed, b) time series analysis where a localized emission is superposed on a background, c) point and extended sources to be detected in time-ordered data where the scanning strategy (for satellites like Planck) is affected by the well-known $1/f$ noise. In the 2D case, we mention as typical cases: a) cleaning of images to detect typical astrophysical sources on a white noise background, b) the detection/extraction of point sources and extended sources (clusters of galaxies) on microwave/IR maps where the background is dominated by white noise or intrinsic CMB signal or galactic emission. In

the 3D case, we remark as an example: a) the detection of clusters and large-scale structure in 3D catalogs.

The classical treatment to remove the background has been *filtering*. Low and high-pass filters reduced the contribution from the high and low frequencies present in the spectrum or image. In general, this process is extremely inefficient in dealing with very localized sources. The reason for that is that a very localized source can be decomposed in Fourier space but many waves are needed (infinite for a delta distribution!), so if low/high-pass filters are applied then at the end many artefacts (rings in 2D images) usually appear surrounding the sources.

A very important application of these principles is the detection of sources in two-dimensional astronomical images. Several packages are commonly used for this task, such as DAOfind (Stetson 1992), used to find stellar objects in astronomical images, and SExtractor (Bertin & Arnouts 1996). When trying to detect sources, two different problems have to be solved: first, it is necessary to take account for the background variations across the image. In addition, if instrumental noise (i. e. white noise) appears, it should be removed as far as possible in order to increase the SNR ratio. SExtractor estimates the local background on a grid of size set by the user and then approximates it by a low-order polynomial. Once such a background is defined the detection of sources at a certain level can be established through sets of connected pixels above certain threshold. DAOfind implicitly assumes that the background is smooth and its characteristic scale is very much larger than the scale of the stars. But in the case where the characteristic scale of variation of the background is approximately the scale of the structures the previous schemes obviously fail. An incorrect estimation of the background leads to a biased estimation of the amplitude of the sources. An example is a typical image of the cosmic microwave background radiation (CMB) at a resolution of several arcmin. If the intrinsic signal is due to a cold dark matter

model (i. e. the characteristic scale of the background is of the order of ~ 10 arcmin), then the separation of point sources with the same characteristic scale becomes a very difficult task.

To deal with the instrumental noise the traditional procedure is filtering (e. g. Gaussian window). DAOfind filters with an elliptical Gaussian that mimics the stellar psf and then it performs the detection looking for peaks above certain threshold. SExtractor includes the possibility of filtering with several kind of filters (Top Hat, Gaussian, Mexican Hat and even self-made filters for every particular situation). An obvious advantage of this procedures (background estimation plus filtering) is that no a priori information on the structures is needed. A serious drawback is that the choice of the filter will have a great influence on the final result. The choice of filter depends on many factors, including in most cases personal preferences.

In this context, it is necessary to find a systematic way to determine the optimal filter for every case.

Other methods have been used to separate different components given several microwave maps: Wiener filtering (WF) and maximum entropy methods (MEM). Regarding point sources, WF assumes a mean spectral flux dependence together with other assumptions for the other components (Tegmark & Efstathiou, 1996; Bouchet et al. 1999) whereas MEM assumes that the point sources are distributed like noise (Hobson et al. 1999). These methods are powerful regarding extended sources like clusters of galaxies because they use the concrete spectral dependence for the Sunyaev-Zeldovich effect. It is clear that the unknown spectral dependence for point sources remark the inefficiency of the previous methods.

A possible solution to overcome this problem came with the usage of *wavelets*. These are localized bases that allow a representation of a local object due, in general, to their

basic properties: spatial and frequency localization (as opposed to the trigonometric functions appearing in Fourier decomposition). We remark at this point the success of such a technique dealing with simulated microwave maps: the "Mexican Hat" wavelet can be used in a nice way (no extra assumptions about the background) to detect/extract point sources and create a simulated catalog (Cayón et al. 2000, Vielva et al. 2000). Two advantages emerge: on the one hand, one localizes the structures associated to the maxima of the wavelet coefficients and, what is more remarkable, we gain in the detection (as compared to real space) due to the amplification effect because at the scale of the source the background is not contributing to the dispersion. One relevant question concerns the possibility to find *optimal* filters. Tegmark & Oliveira-Costa (1998) introduced a filter that minimizes the variance in the map. With this method one can identify a big number of point sources in CMB maps. However, they failed to introduce the appropriate constraints in the minimization problem, i. e. the fact that we have a maximum at the source position at the scale defined by the source in order not to have spurious identifications.

Thus, this type of analysis lead us to the following questions: is there an optimal filter (or better pseudo-filter) given the source profile and the power spectrum of the background?, Is the "Mexican Hat" wavelet the optimal pseudo-filter dealing with point sources? In order to answer these questions, we will assume that the sources can be approximated by structures with central symmetry given by a profile $\tau(x)$, $x \equiv |\vec{x}|$ with a characteristic scale (e.g. a single maximum at its center and rapid decay at large distances). If the nD image contains different types of sources, a number of pseudo-filters adapted to each profile must be used to detect them. A possible generalization to include more general profiles is under study. The background will be modelled by a homogeneous and isotropic random field given in terms of the power spectrum $P(q)$, $q \equiv |\vec{q}|$. In particular, we shall explore a scale-free spectrum $P(q) \propto q^{-\gamma}$ that includes the cases of white noise ($\gamma = 0$), $1/f$ noise ($\gamma = 1$), etc. Moreover, any spectrum of physical interest often can be locally

approximated by a power-law. If the characteristics of the noise are not known a priori it can be always estimated directly from the nD image. We consider the n-dimensional case and make special emphasis on the analysis of spectra ($n = 1$), 2D images ($n = 2$) and 3D volumes ($n = 3$). In all the calculations we assume that the overlapping by nearby sources is negligible and also that their contribution to the total power spectrum is also negligible. All of this is a very good approximation at least above a certain flux level.

An analytical approach to get the optimal pseudo-filter is presented in section 2. Section 3 deals with properties of the optimal pseudo-filters on real space. Section 4 introduces the concepts of detection level and gain. Sections 5 and 6 are dedicated to the important cases of sources with profiles described by a Gaussian and an exponential, respectively. An example of the performance of optimal pseudo-filters applied to simulated one-dimensional data is presented in section 7. Section 8 deals with the extraction of the sources. Conclusions are summarized in section 9.

2. Optimal pseudo-filter

Let us consider an n-dimensional ($n = 1, 2, 3$) image with data values defined by

$$y(\vec{x}) = s(x) + n(\vec{x}), \tag{1}$$

where x is the spatial coordinate (in the 1D case x can be also time, when we are dealing with time-ordered data sets) and $s(x), x \equiv |\vec{x}|$, represents a source with central symmetry placed at the origin with a characteristic scale (e. g. a single maximum at its center and rapid decay at large distances) and $n(\vec{x})$ a homogeneous & isotropic background (random field) with mean value $\langle n(\vec{x}) \rangle = 0$ and characterized by the power spectrum $P(q), q \equiv |\vec{q}|$ (this can represent instrumental noise and/or a real background),

$$\langle n(\vec{q})n^*(\vec{q}') \rangle = P(q)\delta^n(\vec{q} - \vec{q}'), \quad (2)$$

where $n(\vec{q})$ is the nD Fourier transform ($n(\vec{q}) = (2\pi)^{-n/2} \int d\vec{x} e^{-i\vec{q}\vec{x}} n(\vec{x})$), symbol n^* represents the complex conjugate of n , \vec{q} is the wave vector and δ^n is the nD Dirac distribution.

Let us introduce a spherical (centrally symmetric) pseudo-filter, $\Psi(\vec{x}; \vec{b}, R)$, dependent on $n + 1$ parameters .

$$\Psi(\vec{x}; R, \vec{b}) = \frac{1}{R^n} \psi\left(\frac{|\vec{x} - \vec{b}|}{R}\right), \quad (3)$$

where \vec{b} defines a translation whereas R defines a scaling. Then, we define the pseudo-filtered field $w(R, \vec{b})$

$$w(R, \vec{b}) = \int d\vec{x} y(\vec{x}) \Psi(\vec{x}; \vec{b}, R). \quad (4)$$

We do not assume a priori the positiveness of Ψ . The previous convolution can be written as a product in Fourier space, in the form

$$w(R, \vec{b}) = \int d\vec{q} e^{-i\vec{q}\vec{b}} y(\vec{q}) \psi(Rq). \quad (5)$$

where $y(\vec{q})$ and $\psi(q)$ are the Fourier transforms of $y(\vec{x})$ and $\psi(\vec{x})$, respectively. Because of the central symmetry assumed for the pseudo-filter, $\psi(q)$ depends only on the modulus of \vec{q} . A simple calculation -taking into account eqs. (1) and (2)- gives the average at the origin $\vec{b} = 0$, $\langle w(R, \vec{0}) \rangle$, and the variance, $\sigma_w^2(R) = \langle w^2(R, \vec{b}) \rangle - \langle w(R, \vec{b}) \rangle^2$, of the pseudo-filtered field

$$\langle w(R, \vec{0}) \rangle = \alpha \int dq q^{n-1} s(q) \psi(Rq), \quad \sigma_w^2(R) = \alpha \int dq q^{n-1} P(q) \psi^2(Rq). \quad (6)$$

where $q = |\vec{q}|$, $\alpha = 2, 2\pi, 4\pi$ for $n = 1, 2, 3$, respectively, (for n -dimensions $\alpha = 2\pi^{n/2}\Gamma^{-1}(n/2)$) and the limits in the integrals go from 0 to ∞ .

Now, we are going to express the conditions in order to obtain an optimal pseudo-filter for the detection of the source $s(x)$ at the origin. One basic idea is to find a pseudo-filter such that when the original image is filtered with a scale $R_o \approx R_s$ -being R_s the characteristic scale of the source- one obtains the maximum *detection level* \mathcal{D}_w

$$\mathcal{D}_w \equiv \frac{\langle w(R, \vec{0}) \rangle}{\sigma_w(R)}. \quad (7)$$

Taking into account the fact that the source is characterized by a single scale R_s , other basic idea is to generate a filter giving the maximum contribution at the center of the source at a filtering scale $R_o \approx R_s$. Finally, we would like to estimate directly the amplitude of the source by the previous number. Therefore, taking into account these basic ideas we will introduce from the mathematical point of view the optimal pseudo-filters.

By definition a pseudo-filter will be called *optimal* if the following conditions are satisfied:

i) there exists a scale R_o such that $\langle w(R, \vec{0}) \rangle$ has a maximum at that scale, ii) $\langle w(R_o, \vec{0}) \rangle = s(0) \equiv A$, i. e. $w(R, \vec{0})$ is an unbiased estimator of the amplitude of the source and iii) the variance of $w(R, \vec{b})$ has a minimum at the scale R_o , i. e. we have an efficient estimator. As a by-product, the ratio given by eq. (7) will be maximum. We remark that no other information about the source profile is assumed, so “optimal” must be understood in the previous sense.

By introducing the profile $\tau(x)$ of the source, $s(x) = A\tau(x)$, the condition ii) and the

equation (6) give the constraint

$$\int dq q^{n-1} \tau(q) \psi(R_o q) = \frac{1}{\alpha}, \quad (8)$$

whereas the condition i) gives the constraint

$$\int dq q^{n-1} \tau(q) \psi(R_o q) \left[n + \frac{d \ln \tau}{d \ln q} \right] = 0. \quad (9)$$

So, the problem is reduced to the functional minimization (with respect to ψ) of $\sigma_w^2(R)$ given by equation (6) with the constraints given by equations (8) and (9). This minimization incorporate these constraints through a couple of Lagrangian multipliers. The solution (optimal pseudo-filter) is found to be

$$\tilde{\psi}(q) \equiv \psi(R_o q) = \frac{1}{\alpha} \frac{\tau(q)}{P(q)} \frac{1}{\Delta} [nb + c - (na + b) \frac{d \ln \tau}{d \ln q}], \quad \Delta = ac - b^2, \quad (10)$$

$$a \equiv \int dq q^{n-1} \frac{\tau^2}{P}, \quad b \equiv \int dq q^{n-1} \frac{\tau}{P} \frac{d \tau}{d \ln q}, \quad c \equiv \int dq q^{n-1} \frac{1}{P} \left[\frac{d \tau}{d \ln q} \right]^2. \quad (11)$$

Therefore, we have obtained analytically the functional form of the pseudo-filter (its shape and characteristic scale are associated to the source profile and power spectrum). It is clear that assuming an adimensional dependence $\tau(x/R_s)$, where R_s is the characteristic scale of the source, then such scale will appear explicitly in the form $\tilde{\psi}(qR_s)$. Obviously, we assume all the differentiable and regularity conditions at $q \rightarrow 0, \infty$ for τ and $P(q)$ in order to have finite expressions for a, b, c . Generically, ψ is a pseudo-filter, i. e. it is not positive (filter). Let us remark that if we assume the behavior $\frac{\tau(q)}{P(q)} \rightarrow 0, \frac{1}{P} \frac{d \tau}{d \ln q} \rightarrow 0$ for $q \rightarrow 0$ then $\psi(q) \rightarrow 0$ and Ψ is a "compensated" filter, i. e. $\int d\vec{x} \Psi = 0$. Strictly speaking there is

another condition to be satisfied to get the reconstruction of the image and thus to have a wavelet: $\int dq q^{-1} \psi^2(q) < \infty$ (the admissibility condition).

Taking into account eq. (5) the amplitude will be estimated as

$$A = w(R_o, \vec{0}) = \int d\vec{q} y(\vec{q}) \tilde{\psi}(q), \quad (12)$$

where $\tilde{\psi}$ is given by eq.(10).

3. Optimal pseudo-filter on real space

The equation (10) can be written on real space as follows

$$\Psi(\vec{x}; R_o, \vec{0}) = \frac{1}{R_o^n} \psi\left(\frac{|\vec{x}|}{R_o}\right) = \frac{1}{\alpha \Delta} [(nb + c)F(x) - (na + b)G(x)], \quad (13)$$

where F and G are the inverse Fourier transform of $\frac{\tau}{P}$ and $\frac{1}{P} \frac{d\tau}{d \ln q}$, respectively.

For a flat background, i. e. $P = \text{constant}$, and assuming the behaviour $q^{\frac{1}{2}(n+1)} \tau \rightarrow 0$ when $q \rightarrow \infty$, one obtains

$$b = -\frac{n}{2}a, \quad F = \frac{\tau(x)}{P}, \quad G(x) = -\frac{1}{P} \left[n\tau + \frac{d\tau}{d \ln x} \right]. \quad (14)$$

If we also assume a Gaussian profile, i. e. $\tau = e^{-\frac{x^2}{2\theta^2}}$, one finds the pseudo-filter

$$\Psi(\vec{x}; \theta, \vec{0}) = \frac{1}{\pi^{n/2} \theta^n} e^{-\frac{x^2}{2\theta^2}} \left[1 + \frac{n}{2} - \left(\frac{x}{\theta} \right)^2 \right], \quad (15)$$

that is a useful formula to be used for the detection of nD-Gaussian structures on nD-images (e.g. spectra, 2D images or 3D volumes) on a flat background.

On the other hand, if one assumes a Gaussian profile but a non-flat spectrum one can easily find

$$\Psi(\vec{x}; \theta, \vec{0}) = \frac{1}{\alpha\Delta} [(nb + c)F(x) - (na + b)\theta^2 \nabla^2 F(x)], \quad (16)$$

being $F(x)$ the Fourier transform of $\frac{\tau}{P}$.

4. Detection level, gain and reliability

Taking into account the previous expression (10), one can calculate the detection level (see equation(7))

$$\mathcal{D}_w = A \left[\frac{\alpha\Delta}{n^2a + 2nb + c} \right]^{1/2}. \quad (17)$$

On the other hand, we can calculate the dispersion of the field $n(\vec{x})$

$$\sigma_b^2 = \alpha \int dq q^{n-1} P(q), \quad (18)$$

that allows to define a detection level on real space as

$$\mathcal{D} = \frac{A}{\sigma_b}. \quad (19)$$

The *gain* going from real space to pseudo-filter space is defined by

$$g \equiv \frac{\mathcal{D}_w}{\mathcal{D}} = \frac{\sigma_b}{\sigma_w(R_o)}. \quad (20)$$

If the background has a characteristic scale different from the scale of the structures (sources) to be detected, it is obvious that $\sigma_w < \sigma_b$, so that we have a real gain going from real space to pseudo-filter space.

The identification of sources as peaks above a high threshold (e. g. $3\sigma_w$) in pseudo-filter space gives a low probability of false detections (reliability) because if the background has a characteristic scale different from the sources then everything detected with our method is real, but if both scales are comparable one can give an estimate based on the fluctuations of the background. For instance, in the case of a Gaussian background, false detections above $3\sigma_w$, due to the Gaussian background, appear with a probability $\simeq 1.5 \times 10^{-3}$ based on the formula

$$Pr(w > A) = \frac{1}{2} \operatorname{erfc}\left(\frac{A}{2^{1/2}\sigma_w}\right). \quad (21)$$

To select the false detections from the real ones one can study the pseudo-filter profile nearby any real source (see the last paragraph of section 8).

Regarding the completeness (i. e. how many real sources we miss with our method), this is a complicated topic because the background can slightly modify the location of the peaks and their amplitude. We will address this problem via numerical simulations (see section 7).

5. Gaussian source on a background

In many physical applications the standard response of the instruments can be approximated by a Gaussian function. In particular, the Point Spread Function (PSF) for many telescopes is of Gaussian type. Other more specific applications are related to the Cosmic Microwave Background (CMB), where the antennas used are well approximated by

a Gaussian. Dealing with absorption systems associated to QSOs, if the absorption line is not saturated and is dominated by thermal motions, then the line is usually approximated by an inverted Gaussian inserted in a continuum plus noise.

Let us assume that the source and the background can be represented by

$$s(x) = A e^{-\frac{x^2}{2\theta^2}}, \quad P = Dq^{-\gamma}. \quad (22)$$

The structure to be detected could have an intrinsic Gaussian profile or it could be a point source in n -dimensions observed with an instrument that can be modelled through a Gaussian pattern with a beam size θ . The background can be described by a scale-free spectrum. In this case: $\tau(q) = \theta^n e^{-\frac{1}{2}(q\theta)^2}$, and equations (11) give

$$a = \frac{\theta^{n-\gamma}}{2D} \Gamma(m), \quad b = -ma, \quad c = m(1+m)a, \quad m \equiv \frac{n+\gamma}{2} \quad (23)$$

and the pseudo-filter is

$$\tilde{\psi}(q) = \frac{1}{\alpha \Gamma(m)} (q\theta)^\gamma e^{-\frac{1}{2}(q\theta)^2} \left[2 + \gamma - n + \frac{n-\gamma}{m} (q\theta)^2 \right]. \quad (24)$$

Taking into account the q behaviour in this formula, one obtains a compensated filter (i. e. $\psi(q=0) = 0$) if $\gamma > 0$ or $\gamma = n - 2$. In figure 1 appear the optimal pseudo-filters for the 1D, 2D and 3D cases, respectively, and scale-free power spectrum with indexes $\gamma = 0, 1, 2, 3$. There is a degeneration in the case $n = 3$, where the $\gamma = 1$ line overlaps with the $\gamma = 3$ line, and in the case $n = 2$, where the $\gamma = 0$ line overlaps with the $\gamma = 2$ one. This degeneration can be deduced directly from equation (23).

On the other hand, the detection level in pseudo-filter space is given by equation (18)

$$\mathcal{D}_w = A \left[\frac{2\alpha}{D} \frac{\Gamma(1+m)}{4m + (n-\gamma)^2} \right]^{1/2} \theta^{\frac{n-\gamma}{2}}. \quad (25)$$

Finally, it is interesting to remark that the cases $\gamma = n$ and $\gamma = n - 2$ give the same pseudo-filter

$$\tilde{\psi}(q) = \frac{2}{\alpha \Gamma(n)} (q\theta)^n e^{-\frac{x^2}{2\theta^2}}. \quad (26)$$

Therefore, in the cases $n = 2, \gamma = 0, 2$ the Mexican Hat is found to be the optimal pseudo-filter. This justify the use of this wavelet to detect point sources in CMB maps (Cayón 2000, Vielva 2000).

a) Gaussian source and white noise:

This subcase corresponds to $P = \text{constant}$ or $\gamma = 0$, and the pseudo-filter is

$$\tilde{\psi}(q) = \frac{2}{\alpha \Gamma(\frac{n}{2})} e^{-\frac{1}{2}(q\theta)^2} \left[1 - \frac{n}{2} + (q\theta)^2 \right], \quad (27)$$

that gives a pseudo-filter for the analysis in the different dimensions except for $n = 2$ that gives the Mexican Hat wavelet ($\psi = \frac{1}{\pi} e^{-\frac{1}{2}(q\theta)^2} (q\theta)^2$).

The detection level is given by equation (24):

$$\mathcal{D}_w = A \left[\frac{\alpha \Gamma(\frac{n}{2})}{2D(1 + \frac{n}{2})} \right]^{1/2} \theta^{n/2}. \quad (28)$$

We have calculated the contribution of sources to the power spectrum in order to estimate their influence in calculating the pseudo-filter. We arrive to the conclusion that if the signal/noise ratio (i. e. dispersion associated to the sources over dispersion associated

to the background) is $\frac{\sigma_s}{\sigma_b} < 0.6(\frac{l_p}{\theta})$, being l_p the pixel scale and θ the width of the source, then the extra contribution to the pseudo-filter coefficients a, b, c is less than a 10%.

b) Gaussian source and $1/f$ noise:

Let us assume a source with a Gaussian profile and a background represented by $1/f$ noise, i.e. $P = Dq^{-1}$ or $\gamma = 1$. In this case: $\tau(q) = \theta^n e^{-\frac{1}{2}(q\theta)^2}$, and equation (22) gives the pseudo-filter

$$\tilde{\psi}(q) = \frac{1}{\alpha \Gamma(\frac{n+1}{2})} e^{-\frac{1}{2}(q\theta)^2} (q\theta) [3 - n + 2\frac{n-1}{n+1}(q\theta)^2], \quad (29)$$

For instance, in the case $n = 1$ one has the wavelet $\psi = (q\theta)e^{-\frac{1}{2}(q\theta)^2}$, that is the optimal pseudo-filter to be used to detect a Gaussian signal on 1D spectra. In this case the detection level is (see equation(24)) $\mathcal{D}_w = AD^{-1/2}$.

6. Exponential source on a background

Typical example in astrophysics is the exponential disk associated to spiral galaxies. Another interesting application could be in some areas of physics where the profile expected for the signal associated to the detection of some particles could be of exponential type.

Let us assume that the source and background can be represented by

$$s(x) = e^{-\frac{x}{\lambda}}, \quad P = Dq^{-\gamma}. \quad (30)$$

In this case:

$$\tau(q) = \beta \lambda^n [1 + (q\lambda)^2]^{-\frac{n+1}{2}}, \quad (31)$$

where $\beta = (\frac{2}{\pi})^{1/2}, 1, 2(\frac{2}{\pi})^{1/2}$ for $n = 1, 2, 3$, respectively, and equations (11) give

$$a = \frac{\beta^2 \lambda^{n-\gamma}}{2D} \frac{\Gamma(m)\Gamma(1 + \frac{n-\gamma}{2})}{\Gamma(n+1)}, \quad (32)$$

$$b = -ma, \quad c = \frac{n+1}{n+2} m(1+m)a, \quad m \equiv \frac{n+\gamma}{2} \quad (33)$$

and the pseudo-filter is

$$\tilde{\psi}(q) = \frac{2L}{\alpha\beta} (q\lambda)^\gamma [1 + (q\lambda)^2]^{-\frac{n+1}{2}} \left[1 + \frac{\gamma-n}{2} (n+1) + M \frac{(q\lambda)^2}{1 + (q\lambda)^2} \right], \quad (34)$$

$$L \equiv \frac{\Gamma(n+1)}{\Gamma(\frac{n+\gamma}{2})\Gamma(2 + \frac{n-\gamma}{2})}, \quad M \equiv \frac{n-\gamma}{n+\gamma} (n+1)(n+2). \quad (35)$$

In figure 2 appear the optimal pseudo-filters for the 1D, 2D and 3D cases, respectively, and power spectrum with indexes $\gamma = 0, 1, 2, 3$. The filter profiles are more extended than in the Gaussian source, as one can expect from the more gentle fall of the exponential source.

An interesting case is $\gamma = n$, then the pseudo-filter is

$$\tilde{\psi}(q) = \frac{2n}{\alpha\beta} (q\lambda)^n [1 + (q\lambda)^2]^{-\frac{n+1}{2}}. \quad (36)$$

a) Exponential source and white noise:

For this subcase $\gamma = 0$, then equation (36) leads to the pseudo-filter

$$\tilde{\psi}(q) = \frac{2}{\alpha\beta} \frac{\Gamma(n+1)}{\Gamma(\frac{n}{2})\Gamma(2 + \frac{n}{2})} [1 + (q\lambda)^2]^{-\frac{n+1}{2}} \left[1 + \frac{n}{2} (n+1) + \frac{(n+1)(n+2)(q\lambda)^2}{1 + (q\lambda)^2} \right]. \quad (37)$$

For instance, for an exponential structure to be optimally detected in a 1D spectrum, we must use

$$\tilde{\psi}(q) = \frac{8}{(2\pi)^{1/2}} \frac{(q\lambda)^2}{[1 + (q\lambda)^2]^2}. \quad (38)$$

b) Exponential source and $1/f$ noise:

An interesting case is $n = \gamma = 1$, equation (36) gives

$$\tilde{\psi}(q) = \left(\frac{\pi}{2}\right)^{1/2} \frac{(q\lambda)}{1 + (q\lambda)^2}. \quad (39)$$

7. Simulations of one-dimensional Gaussian sources and $1/f$ noise

In order to test some of the ideas proposed in previous sections, we simulated the case of one-dimensional Gaussian sources on a background. The kind of background simulated is the well-known $1/f$ noise. This kind of noise appears very often in many devices in experimental physics. Further simulations with 2-dimensional data and realistic realizations of noise will be carried on in future work.

For the sake of simplicity, all the simulated sources have the same amplitude and that is set to be 1 (in arbitrary units). 100 of these sources were deployed over a 32768 pixel ‘field’. The number of sources and the size of the field were selected in order to have enough sources for statistical studies, to avoid (as far as possible) the overlapping of the sources and to minimize the contribution of the sources to the total dispersion of the simulations. The width of the Gaussian profiles was chosen to be $\theta \simeq 3\theta_p$: this is the case for a pixel of $1.5'$ and a Gaussian with a FWHM of $5'$. Noise was added so that the signal-to-noise ratio of the sources, defined as the ratio A/σ_b (where A is the amplitude of the source and σ_b is the standard deviation of the noise), assumes values of 2, 3, 4 and 5. Finally, the optimal filter, given by eq. (29) with $n = 1$, was applied to the image.

To compare with a more traditional filtering scheme, we filtered the images also with a Gaussian of width equal to θ and a Mexican Hat wavelet of width equal to θ . This is a rather naive usage of the Mexican Hat wavelet and the Gaussian source because the optimal width for these filters in the general case is not the source scale (Cayón 2000, Vielva 2000), but it serves us well because what we intend is to compare how do filters work when we have no further information about the data (i.e., the optimal scale, which is different for each background). The result of these simulations is shown in tables 1 and 2. Table 1 refers to the original simulations. It shows the original signal-to-noise ratio of each simulation as well as statistical quantities of interest such as the dispersion of the map and the mean amplitude of the sources in it. Finally, it shows the number of sources directly detected from the simulations above 3σ and 5σ thresholds and the number of spurious detections above these thresholds. As expected, only a few sources are detected, except for the most favorable cases (high original signal-to-noise ratio and low detection threshold). The small bias in the mean measured amplitude is due to pixelization effects. Table 2 refers to the simulations in table 1 after filtering with a Gaussian of width θ , a Mexican Hat wavelet of width θ and the optimal pseudo-filter. Each row in table 2 corresponds to the same row in table 1.

A Gaussian filter smoothes the image, removing small-scale noise. It also smoothes the source peaks, thus lowering the amplitude of detected sources. For the case of $1/f$ noise the dominant fluctuations appear at large scales and are not affected by the Gaussian filter. The large-scale features may contribute to contamination in two different ways: they can conceal sources in large ‘valleys’ and can produce spurious peaks. None of these effects can be avoided with a Gaussian filter. On the other hand, the smoothing effect of the Gaussian filter takes place normally and lowers the amplitude of the sources. Therefore, the number of true detections is smaller than in the non-filtered image, and the spurious detections are not removed even in the highest s/n case. The gains, indicated in column 6, clearly reflects

this situation ($g < 1$).

The Mexican Hat wavelet has a better performance under $1/f$ conditions. The Mexican Hat removes large-scale fluctuations, allowing the 'hidden' sources to arise above the detection threshold. For example, in the case of original $s/n = 2.95$ there were 47 sources above the 3σ level and only 1 above the 5σ level. After filtering with the Mexican Hat, there are 93 detections above 3σ level and 64 above the 5σ level, a significant improvement. The number of spurious sources remains almost untouched.

The optimal pseudo-filter also deals with the large-scale structure. It is constructed to enhance all fluctuations in the source scale, while removing fluctuations that arise in other scales. In addition, it is required to be unbiased with respect to the amplitude. In practice, the amplitude is slightly underestimated due to the propagation of pixelization effects. This small bias is lower than a 10% and can be calibrated in any case. In the $1/f$ case the number of true detections is higher than in the Mexican Hat case and the number of spurious sources is comparable or slightly reduced. Only in the case of low initial signal-to-noise ratio the number of spurious detections is greater. This is due to the fact that this pseudo-filter enhances all fluctuations in the source scale. Future work will take care of this weakness of the method including more information about the shape of the sources. For the case of initial signal-to-noise ratio of 2.95 we find 94 sources (of 100) and 8 spurious detections (a reliability close to 10%) above the 3σ level, a result very similar to the obtained with the Mexican Hat. Above the 5σ detection level the optimal pseudo-filter finds 79 sources where the Mexican Hat found only 64. The number of spurious sources have not increased significantly (from 4 to 5). For higher initial signal-to-noise ratios the completeness and reliability quickly improve.

The gain obtained with the optimal pseudo-filter is greater than the one obtained with the Mexican Hat. It can be analytically calculated, using eqs. (6,17) for the pseudo-filter

and its equivalents for the Mexican Hat, leading to:

$$g_{op}/g_{mh} = \left[\frac{4}{\pi} \frac{\Gamma(\frac{1+\gamma}{2})\Gamma(\frac{5-\gamma}{2})}{1 + \frac{(1-\gamma)^2}{2(1+\gamma)}} \right]^{1/2} \quad (40)$$

for the one-dimensional case. This formula holds while $\gamma \leq 1$. According to equation (40), the ratio g_{op}/g_{mh} is 1.41 in the case $\gamma = 0$ and 1.13 in the case $\gamma = 1$. The mean observed ratio in the simulations is 1.31 and 1.08 respectively, and fits well with our expectations. As a conclusion we have that optimal filter gives higher gains than the classical Mexican Hat filter.

In figure 3 ^{E1} an example of the simulations is shown. In the top panel there is a 500 pixel wide subsection of the $s/n = 3$, $\gamma = 1$ simulation. This subsection corresponds to a region in which the large-scale noise has a positive value. Four sources are present in this area, all of them arising above the 3σ level (indicated with a dotted line). The position of the sources are marked with an asterisk in the lower panel. Additionally, there are three peaks, corresponding to background fluctuations, that arise above the 3σ level. The second panel from the top shows the image after filtering with the optimal filter. The large-scale features have been removed and also the small-scale noise is reduced. The sources have been amplified with respect to the original map and now all of them reach the 3σ level but the spurious peaks have been removed. The amplitudes of the sources remain unbiased and close to the true value of 1. In the third panel from the top there is the image after filtering with a Gaussian. The whole image has been smoothed and now one of the sources barely reaches the 3σ level. The large-scale fluctuations remain untouched and all the spurious peaks remain in the filtered image. In the bottom panel we see the image after filtering with the Mexican Hat. The large-scale fluctuations are also removed as well as the small-scale

^{E1}NOTE TO EDITOR: Figure 3 is a landscape-styled figure.

noise, as in the case of the optimal pseudo-filter. Nonetheless, the gain is lower and only three of the sources reach the 3σ level. Additionally, it is found that the small-scale noise removal is less efficient in the case of the Mexican Hat.

8. Extraction of sources

The optimal pseudo-filter gives the position and an unbiased estimator of the amplitude of the source. We propose to make the extraction of the source on real space, i. e. one subtracts the function $A\tau(|\vec{x} - \vec{x}_o|)$, being τ the given profile and A the estimated amplitude, at the position of the source \vec{x}_o .

From the practical point of view, in order to select the appropriate sources (with a given scale and avoiding to select spurious detections if the background and/or noise are manifest at scales comparable to the sources) we can operate with the optimal pseudo-filter at other different scales R as given by equation (10) but with $\tilde{\psi}(qx), x \equiv R/R_o$. If the scale that gives the maximum do not correspond to the scale we are looking for then this is a spurious source (or another type of source with a different scale). As an additional check, we can calculate the source profile in the pseudo-filter space nearby any real source, e. g. for a Gaussian profile the behaviour around the maximum $R_o \approx R_s$ must be

$$\langle w(R) \rangle = Ax^\gamma \left(\frac{2}{1+x^2} \right)^m \left[1 + \frac{n-\gamma}{2} \frac{1-x^2}{1+x^2} \right], \quad x \equiv \frac{R}{R_o}, \quad m \equiv \frac{n+\gamma}{2}, \quad (41)$$

and an analogous behaviour can be found for the exponential profile. If a detected source do not follows such a behaviour then it would be consider as a false detection and must be deleted from the initial catalog.

9. Conclusions

We have introduced for the first time (as far as the authors know) the concept of *optimal* pseudo-filter to detect/extract spherical sources on a background modelled by a (homogeneous & isotropic) random field characterized by its power spectrum. We have obtained a generic analytical formula that allows to calculate such a pseudo-filter either in Fourier or real space as a function of the source profile and power spectrum of the background. The pseudofilter is an unbiased and efficient estimator of the amplitude of the sources.

We have applied the previous formula to the cases of a Gaussian and an exponential profile and studied scale-free spectra. In particular, we have remarked the interesting cases of white noise and $1/f$ noise. We have calculated the detection level for the physically interesting cases of spectra, images and volumes. For some particular cases, the optimal pseudo-filters are wavelets (e. g. a Gaussian source embedded in white noise in the 2D case).

We have simulated Gaussian sources embedded in a $1/f$ noise in order to see the performance of the optimal filter against the Mexican Hat wavelet. In the last case the gain is lower, the noise removal is less efficient and the number of real detections is smaller. We also remark that filtering with a Gaussian window is not the optimal procedure.

The extraction of the sources identified at a certain scale is proposed to be done directly on real space. At the location of the source \vec{x}_o one subtracts the function $A\tau(|\vec{x} - \vec{x}_o|)$, being τ the given profile.

All the calculations assume that the overlapping of nearby sources is negligible and the contribution of the sources to the background is also negligible. This is a very good approximation in many cases of interest at least above a certain flux level.

We remark the advantages of our method over traditional ones (DAOFind, SExtractor): we do not need to assume a smooth background and/or some filters (e.g. Gaussian) in order to detect the sources. For some astrophysical cases (CMB) the background can be complex so a smooth surface could be not a reasonable assumption. However, we need to assume the profile of the source and statistical properties of the background in order to get the optimal filter. The main advantage of our method is the amplification effect (gain) going to pseudo-filter space.

Generalization of these studies, considering different kind of sources (including non-centrally symmetric ones), are now being undertaken. The applications of this type of methodology is without any doubt of interest not only for Astrophysics/Cosmology but for other sciences.

This work has been supported by the Comision Conjunta Hispano-Norteamericana de Cooperación Científica y Tecnológica ref. 98138, Spanish DGESIC Project no. PB98-0531-c02-01, FEDER project no. 1FD97-1769-c04-01 and the EEC project INTAS-OPEN-97-1992 for partial financial support. J. L. S. acknowledges partial financial support from Spanish MEC and thanks CfPA and Berkeley Astronomy Dept. hospitality during year 1999. D. H. acknowledges a Spanish M.E.C. PhD. scholarship.

REFERENCES

- Cayón, L., Sanz, J. L., Barreiro, R. B., Martínez-González, E., Vielva, P., Toffolatti, L., Silk, J., Diego, J. M. & Argueso, F. 2000, MNRAS, in press.
- Bertin, E., Arnouts, S. 116, A&AS 117, 393.
- Bouchet, F. R. & Gispert, R. 1999, New Astronomy 4, 449.
- Hobson, M. P., Barreiro, R. B., Toffolatti, L., Lasenby, A. N., Sanz, J. L., Jones, A. W. & Bouchet, F. R. 1999, MNRAS 306, 232.
- Stetson, P.B. 1992, *User's Manual for DAOPHOT II*
- Tegmark, M. & Efstathiou, G. 1996, MNRAS 281, 1297.
- Tegmark, M. & de Oliveira-Costa, A. 1998, ApJ 500L, 83.
- Vielva, P., Martínez-González, E., Cayón, L., Diego, J. M., Sanz, J. L. & Toffolatti, L., 2000, in preparation.

Fig. 1.— Optimal filters for a Gaussian source in a background $P(q) = Dq^{-\gamma}$ for $\gamma = 0$ (white noise, solid line), $\gamma = 1$ ($1/f$ noise, dotted line), $\gamma = 2$ (short-dashed line) and $\gamma = 3$ (large-dashed line). One, two and three-dimensional cases are represented. There is a degeneration in the case $n = 3$, where the $\gamma = 1$ line overlaps with the $\gamma = 3$ line, and in the case $n = 2$, where the $\gamma = 0$ line overlaps with the $\gamma = 2$ one.

Fig. 2.— Optimal filters for a exponential source in a background $P(q) = Dq^{-\gamma}$ for $\gamma = 0$ (white noise, solid line), $\gamma = 1$ ($1/f$ noise, dotted line), $\gamma = 2$ (short-dashed line) and $\gamma = 3$ (large-dashed line). One, two and three-dimensional cases are represented. There is a degeneration in the case $n = 3$, where the $\gamma = 0$ line overlaps with the $\gamma = 2$ one.

Fig. 3.— Performances of optimal pseudo-filter, Gaussian filter and Mexican Hat wavelet. In the top panel a portion of a 32768-pixel one-dimensional simulation is shown. In the simulation there are present Gaussian sources of width $\theta = 3\theta_p$ and $1/f$ noise. The amplitude of the sources is 1 for all of them. The dispersion of the background is set to be $1/3$. The second (third) panel from the top shows the map after filtering with the optimal pseudo-filter (a Gaussian of width θ). The bottom panel shows the map after filtering with a Mexican Hat of width θ . The positions of the sources are marked with asterisks. In the three cases the 3σ level of the resultant map is shown with an horizontal dotted line.

N	s/n	σ	\bar{A}	s/n_m	$\sigma_{A_{op}}$	$d_{3\sigma}$	$e_{3\sigma}$	$d_{5\sigma}$	$e_{5\sigma}$
1	2	0.5085	1.0670	2.0983	0.5075	19	12	0	0
2	3	0.3454	1.0195	2.9514	0.3432	47	9	1	0
3	4	0.2656	0.9994	3.7629	0.2582	72	6	10	0
4	5	0.2189	0.9890	4.5174	0.2070	90	9	36	1

Table 1:

Note. — Simulations. The first column shows the number of the simulation. The second column indicates the original signal-to-noise ratio, as explained in the text. Column 3 indicates the dispersion of the map. Column 4 shows the mean amplitude of the sources as measured from the map and column 6 indicates the variance of the source amplitudes. This quantity is not equal to the original signal-to-noise ratio (column 2) because the sources contribute to the final dispersion of the image, thus lowering the final, measured signal-to-noise ratio. Column 5 indicates the mean signal-to-noise ratio of the sources (that is, the ratio between the quantities in columns 4 and 3). Columns 7 and 8 indicate the number of sources detected at the 3σ level and the number of 3σ detections that do not correspond to real sources, respectively. Finally, columns 9 and 10 indicate the number of sources detected at the 5σ level and the number of 5σ detections that do not correspond to real sources, respectively.

N	σ_f	\bar{A}_f	s/n_f	σ_{A_f}	g	$d_{3\sigma}$	$e_{3\sigma}$	$d_{5\sigma}$	$e_{5\sigma}$
GAUSSIAN FILTER									
1	0.4816	0.7466	1.5504	0.4822	0.7389	8	3	0	0
2	0.3260	0.7237	2.2204	0.3216	0.7523	27	4	0	0
3	0.2494	0.7134	2.8602	0.2411	0.7601	46	3	0	0
4	0.2045	0.7076	3.4604	0.1929	0.7660	67	2	5	0
MEXICAN HAT									
1	0.2507	1.0070	4.0169	0.2475	1.9144	77	31	17	2
2	0.1798	0.9762	5.4298	0.1665	1.8397	93	16	64	4
3	0.1471	0.9642	6.5548	0.1250	1.7419	96	5	94	4
4	0.1292	0.9587	7.4214	0.1008	1.6429	98	2	98	2
OPTIMAL PSEUDO-FILTER									
1	0.2288	1.0159	4.4408	0.2217	2.1164	87	13	26	2
2	0.1673	0.9919	5.9305	0.1496	2.0094	94	8	79	5
3	0.1394	0.9825	7.0474	0.1131	1.8728	97	3	97	3
4	0.1244	0.9778	7.8592	0.0918	1.7398	99	1	99	1

Table 2:

Note. — Gaussian, Mexican Hat and optimal pseudo-filter results. The first column shows the number of the simulation, the second shows the dispersion of the filtered map, the third column indicates the mean estimated amplitude of the sources, the fourth column shows the mean signal-to-noise ratio of the sources, column 5 indicates the dispersion of the measured amplitudes of the sources, column 6 indicates the gain of the filter (calculated through $(s/n)_{filter}/(s/n)_{original}$) and columns 7 and 8 indicate the number of sources detected at a 3σ threshold and the number of 3σ detections that do not correspond to real sources, respectively. Finally, columns 9 and 10 indicate the number of sources detected at the 5σ level and the number of 5σ detections that do not correspond to real sources, respectively.





